Optimal Trajectories of High-Thrust Aircraft

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Future fighter aircraft may have sufficient thrust to sustain maximum-turn-rate flight at the corner velocity where the limits on the maximum lift coefficient and maximum normal-acceleration load factor are met simultaneously. Unfortunately, the usual necessary optimal control conditions break down on these corner-velocity arcs. This paper presents a set of necessary optimality conditions which must hold when corner velocity arcs are part of an optimal aircraft trajectory. First, these necessary conditions are obtained for a general class of problems with two state-dependent control variable inequality constraints. The resulting conditions are identical to those for optimal control problems with state variable inequality constraints. These necessary conditions then are applied to optimal trajectory problems with high-thrust aircraft. Two sample solutions to the problem of minimum time-to-turn through a specified heading angle are presented to illustrate some of the features of optimal trajectories with sustained maximum-turn-rate corner velocity arcs.

I. Introduction

ON many optimal trajectories of fighter aircraft, a turn at the maximum turn rate is required which, in turn, requires the use of the maximum lift coefficient C_L. Below a given velocity, called the corner velocity, C_L is limited by the aerodynamics of the aircraft (the C_L limit). At velocities above the corner velocity, C_L is limited by the maximum allowable normal acceleration of the aircraft, because of structural or human-factor constraints (the load factor limit). The fastest maximum turn rate occurs when the aircraft uses maximum C_L and flies at the corner velocity where both the C_L load factor limits are met simultaneously. With current fighter aircraft, the maximum available thrust is insufficient to overcome the induced drag due to lift while flying at maximum C_L, so that sustained maximum-turn-rate flight at the corner velocity is not possible. Future fighter aircraft, however, may have sufficient thrust available to overcome this induced drag so that the sustained maximum-turn-rate flight at the corner velocity will be feasible. Flight under these conditions will require, in general, a thrust less than maximum in order to maintain the aircraft velocity at the corner velocity. Unfortunately, the usual necessary conditions for an optimal trajectory breakdown for high-thrust aircraft trajectories on which sustained maximum-turn-rate flight at the corner velocity is required.

The purpose of this paper is to present a set of necessary conditions for analyzing these maximum-turn-rate arcs which require a thrust less than maximum to sustain flight at the corner velocity. These necessary conditions turn out to be identical to the necessary conditions for optimal control problems subject to state variable inequality constraints which are presented in Refs. 1 and 2. In Sec. II a general class of optimal control problems with two state-dependent inequality constraints on one control variable is examined. Optimal trajectories of high-thrust aircraft then are considered in Sec. III, where it is shown that these problems are a special case of the general class of problems that was discussed in Sec. II. In Sec. IV two sample solutions to the problem of minimum time-to-turn through a specified heading angle are presented to illustrate some of the features of optimal aircraft trajectories with sustained maximum-turn-rate areas at the corner velocity. These solutions are obtained by a simple backward integration of the state and costate equations from assumed terminal conditions. Some conclusions then are presented in Sec. V.

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II. General Class of Optimal Control Problems

Consider the class of optimal control problems with the vector state equation and initial conditions

$$\dot{x}=f(x,u,t),\ x(0)=x_0 \tag{1}$$

where x is an n-dimensional state vector and u is an m-dimensional control vector. Terminal conditions are given by

$$\psi[x(t_f),t_f]=0 \tag{2}$$

where ψ is an s-vector. The payoff in a Bolza form is

$$J=F[x(t_f),t_f]+\int_0^{t_f} L(x,u,t)\,dt \tag{3}$$

The distinguishing feature of this class of problems is that one component, say u_1, of the control vector u must satisfy two state-dependent inequality constraints of the form

$$G_1(x,u_1,t)\le 0,\ G_2(x,u_1,t)\le 0 \tag{4}$$

Although the other components of u may be subject to non-state-dependent inequality constraints, these addition constraints need not be considered at this time.

If both of inequalities (4) are not satisfied with equality simultaneously, the standard necessary conditions for an optimal control can be applied.[1] The Hamiltonian for this problem is

$$H=L(x,u,t)+\lambda^T f(x,u,t)+\mu_1 G_1(x,u_1,t)$$
$$+\mu_2 G_2(x,u_1,t) \tag{5}$$

The vector costate differential equation is

$$\dot{\lambda}^T=-H_x=-(L_x+\lambda^T f_x+\mu_1 G_{1x}+\mu_2 G_{2x}) \tag{6}$$

The transversality conditions yield

$$\lambda^T(t_f) = F_x[x(t_f), t_f] + \nu^T \psi_x[x(t_f), t_f] \quad (7)$$

$$H(t_f) = -F_t[x(t_f), t_f] - \nu^T \psi_t[x(t_f), t_f] \quad (8)$$

In these equations λ is an n-dimensional costate vector, ν is an s-dimensional Lagrange multiplier associated with the terminal conditions, and μ_1 and μ_2 are scalar Lagrange multipliers associated with control inequality constraints (4). The conditions on μ_1 and μ_2 are

$$\begin{aligned} &\mu_i = 0 \text{ if } G_i(x,u_1,t) < 0 \\ &\mu_i > 0 \text{ if } G_i(x,u_1,t) = 0 \end{aligned} \qquad i = 1,2 \quad (9)$$

Since both G_1 and G_2 cannot be zero simultaneously in the application of the necessary conditions up to this point, μ_1 and μ_2 cannot be nonzero simultaneously. If either μ_1 or μ_2 is nonzero, it is found uniquely from the equation

$$H_{u_1} = L_{u_1} + \lambda^T f_{u_1} + u_i G_{iu_1} = 0 \quad (10)$$

where $i=1$ or 2, depending on which of inequalities (4) is satisfied with equality. Finally, the control vector u must be chosen to minimize H subject to any control constraints.

The preceding conditions break down if $G_1(x,u_1,t) = 0$ and $G_2(x,u_1,t) = 0$ simultaneously. To see this, note that unique values for μ_1 and μ_2 are required in the costate differential equation. However, the only equation available that contains μ_1 and μ_2 is

$$H_{u_1} = L_{u_1} + \lambda^T f_{u_1} + u_1 G_{1u_1} + \mu_2 G_{u_1} = 0 \quad (11)$$

Therefore, unique solutions for these quantities cannot be found, and a different technique is required to treat this type of constrained arc.

One approach in obtaining necessary optimality conditions for arcs with $G_1(x,u_1,t) = G_2(x,u_1,t) = 0$ is to note that, if these equations are independent in x and u_1, u_1 can be eliminated from these equations, giving a state equality constraint that must be satisfied along these arcs. This constraint has the form

$$S(x,t) = 0 \quad (12)$$

The techniques given in Sec. 3.11 of Ref. 1 now can be applied directly to obtain necessary optimality conditions for this state-constrained arc. Successive differentiation of Eq. (12) gives a set of equations of the following form:

$$S(x,t) = 0, \quad \dot{S}(x,t) = 0, \ldots \quad (13a)$$

$$S^{(q-1)}(x,t) = 0, \quad S^{(q)}(x,u,t) = 0 \quad (13b)$$

Note that the control vector u appears in the qth derivative of $S(x,t)$. Now this arc can be considered to be the result of a state-dependent control constraint

$$S^{(q)}(x,u,t) = 0 \quad (14)$$

with q interior point constraints at the beginning and end of the arc of the form

$$S(x,t) = \dot{S}(x,t) = \ldots = S^{(q-1)}(x,t) = 0 \quad (15)$$

To obtain general conditions for problems with this type of constrained arc, it is convenient to define a new Hamiltonian by

$$\begin{aligned} H = {} & L(x,u,t) + \lambda^T f(x,u,t) + \mu_1 G_1(x,u_1,t) \\ & + \mu_2 G_2(x,u_1,t) + \mu_3 S^{(q)}(x,u,t) \end{aligned} \quad (16)$$

where the multipliers μ_i, $i = 1, 2, 3$ now are defined as follows. If both of the inequalities (4) are not satisfied with equality simultaneously, then $\mu_3 = 0$ and μ_1 or μ_2 is found from Eq. (10). On arcs along which these inequalities are both satisfied with equality, we define $\mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$. The multiplier μ_3 and the control variables u_i, $i = 2, \ldots, m$ then must satisfy Eq. (14) and the following set of equations:

$$H_{u_i} = L_{u_i} + \lambda^T f_{u_i} + \mu_3 S^{(q)}_{u_i} = 0, \quad i = 2, \ldots, m \quad (17)$$

The costate differential equation is the same as before, except for an additional μ_3 term:

$$\dot{\lambda}^T = -H_x = -(L_x + \lambda^T f_x + \mu_1 G_{1x} + \mu_2 G_{2x} + \mu_3 S^{(q)}_x) \quad (18)$$

Because of the interior point constraints given by Eqs. (15), H and λ are, in general, discontinuous at the beginning and end of an arc with $S^{(q)}(x,u,t) = 0$. However, H and λ are not unique on this type of arc, so that these quantities can be defined to be discontinuous at the beginning of the arc and continuous at the end.[2] Let t_1 denote the time of entry into an $S^{(q)}(x,u,t) = 0$ arc and define the interior point constraint of dimension q to be

$$N = [S \dot{S} \ldots S^{(q-1)}]^T = 0 \quad (19)$$

The discontinuity in λ at time t_1 is given by[1]

$$\lambda^T(t_1^-) = \lambda^T(t_1^+) + \pi^T N_x|_{t_1} \quad (20)$$

where π is a q vector of constant Lagrange multipliers that must be found so that Eq. (19) is satisfied at t_1, and the quantities H and λ are continuous at the termination of the $S^{(q)}(x,u,t) = 0$ arc. Similarly, the discontinuity in H at t_1 is

$$H(t_1^-) = H(t_1^+) - \pi^T N_t|_{t_1} \quad (21)$$

The necessary conditions for optimal control problems with state variable inequality constraints that have been used in this analysis actually underspecify the conditions at junctions between arcs with $S(x,t) \neq 0$ and $S(x,t) = 0$. This is discussed in Ref. 3. This reference also gives an example with a state variable inequality constraint which has a solution with an arc on the constraint boundary. This solution satisfies the necessary conditions of Ref. 2 but is not optimal, as is shown using a set of stronger necessary conditions. Thus, care should be taken in interpreting the results just obtained, since the conditions can, in some problems, produce nonoptimal solutions.

To summarize, the optimal control problem considered in this section is defined by Eqs. (1-4, 14, and 15). Necessary conditions for the solution of this class of problems are given by Eqs. (7-9 and 16-21) with the control vector u minimizing H. The conditions on μ_1, μ_2, and μ_3 are given by Eqs. (17) and the discussion following Eq. (16).

III. Optimal Trajectories of High-Thrust Aircraft

In this section it is shown that optimal control problems involving aircraft with high thrust-to-weight ratios are members of the general class of problems considered in Sec. II. The necessary conditions of Sec. II then are applied to these problems assuming a Meyer problem formulation.

The "point mass" aircraft equations of motion are‡

$$\dot{X} = V\cos\psi\cos\gamma, \quad \dot{Y} = V\sin\psi\cos\gamma, \quad \dot{Z} = V\sin\gamma \quad (22a)$$

$$\dot{V} = g[(T-D)/W - \sin\gamma], \quad \dot{\psi} = gL\sin\phi/VW\cos\gamma \quad (22b)$$

‡These point-mass aircraft equations assume a small angle of attack. However, when maximum C_L is used, the angle of attack is not necessarily small, in which case the results obtained using these equations must be viewed as indicative of trends rather than exact solutions.

$$\dot{\gamma}=g(L\cos\phi/W-\cos\gamma)/V \tag{22c}$$

where X and Y are horizontal position coordinates, Z is altitude, V is velocity, ψ is the heading angle, γ is the flight path angle, W is the aircraft weight (which is assumed to be constant here), T is the thrust, D is drag, L is lift, ϕ is the bank angle, and g is the acceleration of gravity. Lift L has the form

$$L=\tfrac{1}{2}\rho(Z)AV^2C_L \tag{23}$$

where $\rho(Z)$ is the air density, A is a reference area, and C_L is the lift coefficient. The drag D is assumed to be a function of V, Z, and C_L and increases monotonically with C_L for fixed V and Z.

The controls are the lift coefficient C_L, the bank angle ϕ, and the thrust T. The thrust must satisfy the constraint

$$T_{\min}\le T\le T_{\max} \tag{24}$$

There are three constraints on C_L. First, there is an upper limit of $C_{L\max}$ due to the aerodynamics of the aircraft. There is also a lower limit of zero. These two constraints can be summarized as

$$0\le C_L\le C_{L\max} \tag{25}$$

There is, in addition, an upper limit on the maximum acceleration of the aircraft normal to the velocity vector which can be expressed as

$$(\tfrac{1}{2})\rho AV^2C_L/W\le n_{\max} \tag{26}$$

where $n_{\max}$ is the maximum allowable normal acceleration in g.

If both the $C_{L\max}$ and $n_{\max}$ limits are met simultaneously, Eqs. (25) and (26) can be solved for the "corner velocity" V_c at any given altitude

$$V_c^2=V^2=2Wn_{\max}/\rho(Z)AC_{L\max} \tag{27}$$

This equation can be written in the same form as Eq. (12):

$$\rho(Z)V^2-2Wn_{\max}/AC_{L\max}=0 \tag{28}$$

This state equality constraint must hold if the aircraft flies with maximum turn rate at the corner velocity. Differentiation of the equation gives

$$(\partial\rho/\partial Z)V^2\dot{Z}+2\rho V\dot{V}=V\{(\partial\rho/\partial Z)V^2\sin\gamma$$
$$+2\rho g[(T-D)/W-\sin\gamma]\}=0 \tag{29}$$

or, since $V>0$,

$$(\partial\rho/\partial Z)V^2\sin\gamma+2\rho g[(T-D)/W-\sin\gamma]=0 \tag{30}$$

This equation contains the thrust T explicitly, thereby allowing us to solve for the value of T required to sustain maximum-turn-rate flight at the corner velocity:

$$T_c=T=D+W[\sin\gamma-(\partial\rho/\partial Z)V^2\sin\gamma/2\rho g] \tag{31}$$

Since T always must satisfy inequality (24), it easily is seen that, if $T_c>T_{\max}$, sustained maximum-turn-rate flight at the corner velocity is not possible, as is the case with current fighter aircraft. Equation (30) is the state-dependent control equality constraint that must hold along a corner velocity arc, and Eq. (28) is the one-dimensional interior-point constraint corresponding to Eqs. (14) and (15), respectively, in Sec. II.

Assuming a Meyer formulation of the optimal control problem, the necessary conditions obtained in the previous section now can be applied. The Hamiltonian is

$$H=V(\lambda_x\cos\gamma\cos\psi+\lambda_y\cos\gamma\sin\psi+\lambda_z\sin\gamma)$$
$$+\lambda_v g[(T-D)/W-\sin\gamma]+\lambda_\psi\rho AVC_L\sin\phi/2W\cos\gamma$$
$$+\lambda_\gamma g(\rho AVC_L\cos\phi/2W-\cos\gamma/V)$$
$$+\mu_2(\rho AV^2C_L/2-n_{\max}W)$$
$$+\mu_3\{(\partial/\partial Z)V^2\sin\gamma+2\rho g[(T-D)/W-\sin\gamma]\} \tag{32}$$

where $\mu_2\neq 0$ only if $C_L<C_{L\max}$, and

$$\rho AV^2C_L/2=n_{\max}W \tag{33}$$

The multiplier μ_3 is nonzero only if Eq. (33) is satisfied and $C_L=C_{L\max}$: (This is the condition for sustained maximum-turn-rate flight at the corner velocity). Since the constraints on thrust and the constraints on C_L given by inequality (25) are not state dependent, they do not effect the costate equations. Therefore, they do not need to be adjoined to H with Lagrange multipliers.

The costate differential equations are

$$\dot{\lambda}_x=\dot{\lambda}_y=0 \tag{34}$$

$$\dot{\lambda}_z=\lambda_v g(\partial D/\partial Z)/W-\lambda_\psi g(\partial\rho/\partial Z)AVC_L\sin\phi/2W\cos\gamma$$
$$-\lambda_\gamma g(\partial\rho/\partial Z)AVC_L\cos\phi/2W-\mu_2(\partial\rho/\partial Z)AV^2C_L/2$$
$$-\mu_3\{(\partial^2\rho/\partial Z^2)V^2\sin\gamma+2g(\partial\rho/\partial Z)[(T-D)/W$$
$$-\sin\gamma]-2\rho g(\partial D/\partial Z)/W\} \tag{35}$$

$$\dot{\lambda}_v=-(\lambda_x\cos\gamma\cos\psi+\lambda_y\cos\gamma\sin\psi+\lambda_z\sin\gamma)$$
$$+\lambda_v g(\partial D/\partial V)/W-\lambda_\psi g\rho AC_L\sin\phi/2W\cos\gamma$$
$$-\lambda_\gamma g(\rho AC_L\cos\phi/2W+\cos\gamma/V^2)-\mu_2\rho AVC_L$$
$$-\mu_3[2(\partial\rho/\partial Z)V\sin\gamma-2\rho g(\partial D/\partial V)/W] \tag{36}$$

$$\dot{\lambda}_\psi=V(\lambda_x\cos\gamma\sin\psi-\lambda_y\cos\gamma\cos\psi) \tag{37}$$

$$\dot{\lambda}_\gamma=V(\lambda_x\sin\gamma\cos\psi+\lambda_y\sin\gamma\sin\psi-\lambda_z\cos\gamma)$$
$$+\lambda_v g\cos\gamma-\lambda_\psi g\rho AVC_L\sin\phi\sin\gamma/2W\cos^2\gamma)$$
$$+\lambda_\gamma g\sin\gamma/V-\mu_3[(\partial\rho/\partial Z)V^2\cos\gamma-2\rho g\cos\gamma] \tag{38}$$

The control variables must minimize H. Thus the optimal bank angle ϕ is given by

$$\sin\phi=-(\lambda_\psi/\cos\gamma)/[(\lambda_\psi/\cos\gamma)^2+\lambda_\gamma^2]^{1/2} \tag{39a}$$

$$\cos\phi=-\lambda_\gamma/[(\lambda_\psi/\cos\gamma)^2+\lambda_\gamma]^{1/2} \tag{39b}$$

If the optimal lift coefficient is interior to the inequality constraints (25) and (26), C_L is found from the solution to the equations

$$\partial H/\partial C_L=-\lambda_v g(\partial D/\partial C_L)/W-g\rho AV[(\lambda_\psi/\cos\gamma)^2$$
$$+\lambda_\gamma^2]^{1/2}/2W=0 \tag{40}$$

and

$$-\lambda_v(\partial^2D/\partial C_L^2)>0 \tag{41}$$

where Eqs. (39) have been substituted for $\sin\phi$ and $\cos\phi$. If inequality (41) is not satisfied or if the solution to Eq. (40)

does not satisfy the inequality constraints on C_L, then the optimal C_L is given by Eq. (33), $C_L = C_{L\max}$, or $C_L = 0$, depending on which value minimizes H and satisfies all the constraints. The optimal thrust is given by

$$T = T_{\max} \text{ if } \lambda_v < 0 \tag{42a}$$

$$T = T_{\min} \text{ if } \lambda_v > 0 \tag{42b}$$

if both the $C_{L\max}$ and $n_{\max}$ constraints are not satisfied with equality simultaneously. If both are satisfied with equality, the expression for the optimal thrust is given by Eq. (31), provided that inequality (24) is satisfied. It can be shown that this is the only condition under which an intermediate thrust level can be optimal; i.e., there are no optimal singular thrust arcs in the conventional sense in this problem.

In the costate differential equations, expressions for μ_2 or μ_3 are needed if either is nonzero. (Recall that both cannot be nonzero simultaneously.) If $\mu_2 \neq 0$, C_L is found from Eq. (33) and μ_2 is found from $\partial H/\partial C_L = 0$ to be

$$\mu_2 = 2\{[\lambda_v g(\partial D/\partial C_L)/W + (g\rho AV/2W)]\,[(\lambda_\psi/\cos\gamma)^2 + \lambda_\gamma^2]^{1/2}\}/\rho AV^2 \tag{43}$$

The multiplier μ_3, which is nonzero only during sustained maximum-turn-rate arcs at the corner velocity, is found from $\partial H/\partial T = 0$ to be

$$\mu_3 = -\lambda_v/2\rho \tag{44}$$

If an optimal corner velocity arc starts at time t_1, a direct application of Eq. (20) to the interior point constraint given by Eq. (28) gives the following discontinuities in λ_z and λ_v at t_1

$$\lambda_z(t_1^-) = \lambda_z(t_1^+) + \pi V^2(\partial\rho/\partial Z) \tag{45}$$

$$\lambda_v(t_1^-) = \lambda_v(t_1^+) + 2\pi\rho V \tag{46}$$

where π is a scalar multiplier here. All of the other costates and the Hamiltonian are continuous at t_1. If this optimal corner velocity arc terminates at time t_2, all of the costate variables and the Hamiltonian are continuous here, as was discussed in Sec. II.

The application of the necessary conditions results in a multiple-point boundary-value problem that must be solved to find candidates for the optimal trajectory. To summarize this multiple-point boundary-value problem, the state and costate differential equations are given by Eqs. (22 and 34-38), respectively, the controls by Eqs. (31 and 39-42), and the multipliers μ_2 and μ_3 by Eqs. (43) and (44) when they are nonzero. Usually the initial states all are specified. At the beginning of a maximum-turn-rate corner velocity arc, Eqs. (28, 45 and 46) must be satisfied, and all of the states, the costates λ_x, λ_y, λ_ψ, and λ_γ, and the Hamiltonian are continuous. At the termination of an optimal corner velocity arc, Eq. (28) must again be satisfied, and all of the states, costates, and Hamiltonian are continuous. At the final time, any specified terminal conditions and the resulting transversality conditions must be satisfied. The multiplier π in Eqs. (45) and (46) must be chosen so that all of these conditions are satisfied.

IV. Minimum Time-to-Turn Problem

To illustrate the characteristics of optimal control problems with the type of constrained arc considered in the previous two sections, here we consider the problem of finding the high-thrust aircraft trajectory that results in the minimum time-to-turn through a specified heading angle. All of the initial states are given, and only the final heading angle ψ_f is specified. All of the other final states are free. The aircraft parameters for this problem are $W = 42{,}000$ lb, $A = 430$ ft^2, $C_{L\max} = 1$, $n_{\max} = 5g$, $T_{\max} = 65{,}270$ lb, and $T_{\min} = -6527$ lb. The expression for drag is

$$D = (\tfrac{1}{2})\rho AV^2(C_{D0} + kC_L^2) \tag{47}$$

where the zero-lift drag coefficient is $C_{D0} = 0.04$, and the induced drag factor is $k = 0.2$. With the exception of $T_{\max}$, which is very high, all of these parameters are typical of current fighter aircraft. The negative value of $T_{\min}$ is attained through the use of speed brakes. The exponential air density expression is, in slugs per cubic foot,

$$\rho = 0.0023769e^{(-Z/23{,}800)} \tag{48}$$

where Z is in feet. The payoff is time or

$$J = t_f \tag{49}$$

The expressions for the state equations, the costate equations, the control variables, and the multipliers μ_2 and μ_3 given in Sec. III are valid here. Applying the transversality conditions given by Eqs. (7) and (8) in Sec. II, we obtain

$$\lambda_x(t_f) = \lambda_y(t_f) = \lambda_z(t_f) = \lambda_v(t_f) = \lambda_\gamma(t_f) = 0 \tag{50}$$

$$H(t_f) = -1 \tag{51}$$

With $\lambda_\gamma(t_f) = 0$ and $\lambda_\psi(t_f) \neq 0$, in general, Eq. (39) gives, at t_f,

$$\cos\phi = 0, \quad \sin\phi = \pm 1 \tag{52}$$

where the positive sign is chosen if $\lambda_\psi(t_f) < 0$ and the negative sign if $\lambda_\psi(t_f) > 0$. A combination of Eqs. (32 and 50-52) yields the following expression for $\lambda_\psi(t_f)$:

$$\lambda_\psi(t_f) = \mp 2W\cos\gamma/g\rho AVC_L|_{t_f} \tag{53}$$

In the specific examples that follow, the upper signs are always chosen in Eqs. (52) and (53).

When a maximum-turn-rate arc at the corner velocity is part of an optimal trajectory for this problem, the trajectory terminates on this arc. Thus we have a three-point boundary-value problem to be solved here. In order to generate representative optimal trajectories from this three-point boundary-value problem, solutions were generated by backward integration from assumed conditions with the transversality conditions satisfied. Switching from the corner velocity arc to a $T = T_{\max}$ or $T = T_{\min}$ arc then is forced at some time t_1. Equating the expressions for the Hamiltonian at t_1^- and t_1^+, and using Eqs. (45) and (46), the following expression for π is obtained

$$\pi = g\lambda_v^+(T^+ - T^-)/[2V\rho g(T^- - D - W\sin\gamma) + WV^3(\partial\rho/\partial Z)\sin\gamma] \tag{54}$$

where the superscript plus indicates that the quantity is evaluated at t_1^+ on the corner velocity arc, and minus indicates that the quantity is evaluated at t_1^- on the $T_{\max}$ or $T_{\min}$ arc. Note that the only quantity evaluated at t^- is the thrust T. For this problem it turns out that the same value of π is obtained for both $T^- = T_{\max}$ and $T^- = T_{\min}$. This value of π always results in $\lambda_v(t_1^-) = 0$. The value of T^- then is determined by

$$T^- = T_{\max} \text{ if } \dot{\lambda}_v > 0 \tag{55a}$$

$$T^- = T_{\min} \text{ if } \dot{\lambda}_v < 0 \tag{55b}$$

The backward integration then is stopped at some arbitrary initial time.

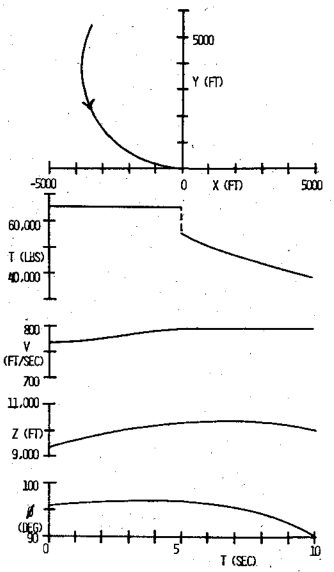

Fig. 1 Results for example problem with thrust sequence $\{T_{max}, T_c\}$.

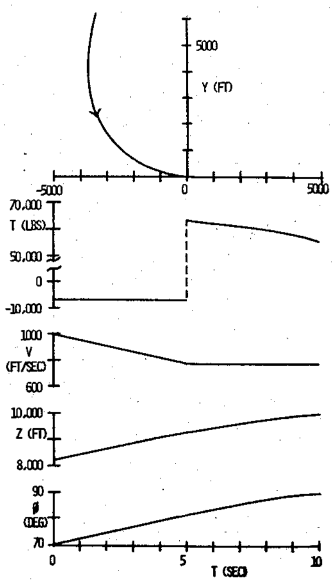

Fig. 2 Results for example problem with thrust sequence $\{T_{min}, T_c\}$.

Figure 1 shows the solution to a typical problem with thrusting sequence $\{T_{max}, T_c\}$. The assumed values of t_I and t_f are $t_I = 5$ sec and $t_f = 10$ sec. The initial states (at $t = 0$) are $X = -3506$ ft, $Y = 5457$ ft, $Z = 9380$ ft, $V = 769$ fps, $\psi = -2.05$ rad, and $\gamma = 0.39$ rad. The resulting final states, at $t = t_f = 10$ sec, are $X = Y = 0$, $Z = 10{,}000$ ft, $V = 791$ fps (the corner velocity at 10,000-ft alt), $\psi = 0$, and $\gamma = -0.20$ rad. The lift coefficient is on the C_{Lmax} limit throughout the flight. Some trends can be noted from Fig. 1. The velocity starts below the corner velocity with a positive flight-path angle γ. In order to quicken the acceleration of the aircraft to the corner velocity, the bank angle ϕ initially is set at about 96° so that a component of lift acts to decrease γ and, hence, increase $\dot{V}$. Once the corner velocity is reached, the thrust is reduced to sustain flight at this condition. This corner velocity thrust decreases between t_I and t_f because the flight-path angle γ decreases from 0.08 rad, where a component of gravity tends to reduce $\dot{V}$, to —0.20 rad, where a component of gravity tends to increase $\dot{V}$, allowing a corresponding reduction in thrust.

Figure 2 presents similar results for a typical problem with a $\{T_{min}, T_c\}$ thrust sequence. Again, $t_I = 5$ sec and $t_f = 10$ sec. The initial states are $X = —3387$ ft, $Y = 6028$ ft, $Z = 8292$ ft, $V = 10{,}037$ fps, $\psi = —1.94$ rad, and $\gamma = 0.20$ rad. The resulting final states are the same as for the previous problem except that $\gamma(t_f) = 0.1$ rad. (It is the final value of γ that determines the optimal thrust sequence in these problems.) The lift coefficient is on the load factor limit throughout this trajectory. Note that $T = T_{min}$ initially is required to reduce the velocity to the corner velocity. This deceleration is aided by using a bank angle less than 90° to increase γ which, in turn, further reduces $\dot{V}$.

Some general trends can be noted on the choice of control required in these minimum time-to-turn problems. The maximum allowable value of C_L always is used. The initial thrust and the initial bank are determined by the initial velocity $V(0)$. If $V(0) > V_c$, the corner velocity, $T(0) = T_{min}$ and ϕ is set slightly less than 90° to decrease γ and thereby further increase V. Once the corner velocity is attained, the thrust is adjusted to sustain flight at this condition and ϕ smoothly approaches 90° as the desired terminal heading is approached.

Conclusions

Sustained maximum-turn-rate arcs at the corner velocity will be of considerable importance for future high-thrust fighter aircraft when the fastest possible turn is required. In the context of optimal trajectories for these aircraft, the corner velocity arcs increase the complexity of the solution to optimal trajectory problems, in that a multipoint boundary-value problem now must be solved rather than a two-point boundary-value problem required for current fighter aircraft. However, by investigating general trends in the solution to these multipoint boundary-value problems, it may be possible to generate near optimal "rules of thumb," as was done with the examples in Sec. IV, to aid the pilot in flying near-optimal maneuvers.

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